Advanced Systems Theory

10/04/2024, Wednesday, 15:00 – 17:00

---

1 Disturbance decoupling with measurement feedback (15 + 10 = 25 pts)

Consider the system

$$\begin{aligned} \dot{x} &= Ax + Bu + Ed \\ y &= Cx \\ z &= Hx \end{aligned}$$

with

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \ E = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, H = \begin{bmatrix} 1 & 1 & a \end{bmatrix}$$

where a is a real number.

(a) Determine all values of a such that the problem of disturbance decoupling (from d to z) by measurement feedback is solvable.

(b) For those values of a, compute a dynamic controller that makes the system from d to z decoupled.

2 Lyapunov equation 10 + 15 = 25 pts

Consider the system

$$\dot{x}(t) = Ax(t) + Bu(t). \qquad (\star)$$

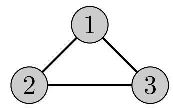
Assume that K, M are matrices such that $M \geqslant 0$, $K > 0$, $A^T K + KA = 0$, and $(MB^T K, A)$ is observable.

(a) Show that all eigenvalues of A are on the imaginary axis.

(b) Show that the feedback $u = Fx$ where $F = -MB^T K$ stabilizes the system $(\star)$.

3 Synchronization (5 + 5 + 10 = 20 pts)

Consider the multi-agent system given by the communication graph G



agent dynamics

$$\dot{x}_i = \begin{bmatrix} 0 & 1 \\ a & a \end{bmatrix} x_i + \begin{bmatrix} 0 \\ 1 \end{bmatrix} z_i,$$

and the diffusive coupling

$$z_i = \begin{bmatrix} 1 & 1 \end{bmatrix} \sum_{j \in N(i)} (x_j - x_i)$$

where $a \in \mathbb{R}$ and $i \in \{1, 2, 3\}$.

(a) Compute the Laplacian L of G.

(b) Verify that $\sigma(L) = \{0, 3\}$.

(c) Determine all values of a such that the system is synchronized?

4 Data-driven control (observability) (5 + 15 = 20 pts)

A linear state/output system of the form

$$\begin{aligned} x(t+1) &= Ax(t) \\ y(t) &= Cx(t) \end{aligned}$$

(or (C, A)) is observable if and only if $\begin{bmatrix} A - \lambda I \\ C \end{bmatrix}$ has full column rank for all $\lambda \in \sigma(A)$. Consider a linear state/output system

$$\begin{aligned} x(t+1) &= A_{\text{true}} x(t) \\ y(t) &= C_{\text{true}} x(t). \end{aligned}$$

Suppose that $(C_{\text{true}}, A_{\text{true}})$ is **unknown** but we harvest the data

$$X = \begin{bmatrix} x(0) & x(1) & \cdots & x(T) \end{bmatrix} \qquad Y = \begin{bmatrix} y(0) & y(1) & \cdots & y(T-1) \end{bmatrix}$$

for some $T > 0$. Define

$$X_- = \begin{bmatrix} x(0) & x(1) & \cdots & x(T-1) \end{bmatrix}, \qquad X_+ = \begin{bmatrix} x(1) & x(2) & \cdots & x(T) \end{bmatrix},$$

and

$$\Sigma(Y, X) = \{(C, A) \mid Y = CX_- \text{ and } X_+ = AX_-\}.$$

Note that $(C_{\text{true}}, A_{\text{true}}) \in \Sigma(Y, X)$.

(a) Show that $\Sigma(Y, X) = \{(C_{\text{true}}, A_{\text{true}})\}$ if and only if X_- has full row rank.

(b) We say that the data (Y, X) are *informative for observability* if (C, A) is observable for all $(C, A) \in \Sigma(Y, X)$. Prove that the data (Y, X) are informative for observability if and only if $\Sigma(Y, X) = \{(C_{\text{true}}, A_{\text{true}})\}$ and $(C_{\text{true}}, A_{\text{true}})$ is observable.

---

10 pts free